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On the Convergence of Pólya's Algorithm

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INTRODUCTION

One of the central questions of Tchebycheff approximation is computing the polynomial of best approximation. The underlying idea of the algorithms of computation is usually approximation of Tchebycheff norm by other norms.

Consider for example the Pólya algorithm. Let $f \in C[0, 1]$, let $p_n(f)_C$ be the algebraic polynomial of degree n of best Tchebycheff approximation to f, and $p_n(f)_q$ $(q \geqslant 1)$ the algebraic polynomial of degree n of best L_q approximation to f. Then as was shown by Pólya [1] $p_n(f)_q$ converges uniformly to $p_n(f)_C$ as $q \to +\infty$. The analogue of this theorem for the de la Vallée–Poussin (or discrete) algorithm was proved by Motzkin and Walsh [2, 3]. Moreover Cheney [4] proved that

$$\| p_n(f)_C - p_n(f)_Y \|_C \leqslant C(n, f)\, \omega_f(|Y|),$$

where $p_n(f)_Y$ is the best Tchebycheff approximation to f on $Y \subset [0, 1]$, $\omega_f(\delta)$ is the modulus of continuity of f and $|Y| = \sup_{x \in [0,1]} \inf_{y \in Y} |x - y|$. Some theorems on uniform convergence of de la Vallée–Poussin algorithm for classes of continuous functions were proved in [5].

In the present paper we shall investigate the rate of convergence of Pólya algorithm. As it was shown by Peetre [6], if $f \in C[0, 1]$ is continuously differentiable then for $q \geqslant q_0$

$$\| p_n(f)_C - p_n(f)_q \|_C \leqslant C(n, f)\, \frac{\ln q}{q}.$$

Our aim is to prove a theorem on convergence of Pólya algorithm for arbitrary $f \in C[0, 1]$. Moreover we shall verify the sharpness of our estimations. At last we give a theorem on uniform convergence of Pólya algorithm.

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In what follows $C_i(\cdots)$ and $q_i(\cdots)$ denote positive constants depending only on quantities specified in the brackets; while C_i and q_i denote positive absolute constants.

Main Theorems

Let $f \in C[0, 1]$. We shall use the following notation

$$\|f\|_C = \max_{x \in [0,1]} |f(x)|; \qquad \|f\|_q = \left(\int_0^1 |f(x)|^q \, dx\right)^{1/q} \qquad (q \geqslant 1);$$

$$\omega_f(\delta) = \sup_{\substack{x_1, x_2 \in [0,1] \\ |x_1 - x_2| \leqslant \delta}} |f(x_1) - f(x_2)|;$$

$p_n(f)_C$ and $p_n(f)_q$ are algebraic polynomials of order at most n of best approximation in C and L_q norm respectively ($n \in \mathbb{Z}_+$). Further define $E_1 = E_1(q)$ as the unique solution of the equation

$$\frac{1}{E_1} = \omega_f(e^{-q/E_1}) \qquad (E_1 > 0; q \geqslant 1). \tag{1}$$

It can be easily verified that $E_1(q)$ monotonously tends to infinity as $q \to +\infty$ and $q/E_1(q) > C \ln q$ for $q \geqslant q_0$.

Theorem 1. *Let $f \in C[0, 1]$. Then for any $q \geqslant 1$ and $n \in \mathbb{Z}_+$,*

$$\|p_n(f)_C - p_n(f)_q\|_C \leqslant \frac{C_0(n, f)}{E_1(q)}. \tag{2}$$

Let us consider some concrete cases. If $\omega_f(\delta) \leqslant \delta^\alpha$ $(0 < \alpha \leqslant 1;\ 0 < \delta \leqslant 1)$, then $E_1(q) \geqslant \alpha q/\ln q$ $(q \geqslant e^\alpha)$. If $\omega_f(\delta) \leqslant \exp[-\alpha \ln^b(1/\delta)]$ $(0 < b < 1, \alpha > 0)$, then $E_1(q) \geqslant \alpha^{1/b} q/\ln^{1/b} q$ $(q \geqslant e^\alpha)$. For $\omega_f(\delta) \leqslant 1/\ln^a(1/\delta)$ $(a > 0)$ we have $E_1(q) \geqslant q^{a/(a+1)}$ $(q \geqslant 1)$.

It turned out that estimation (2) is in general the best possible. We shall need some additional definitions. Let W be the set of all moduli of continuity of continuous functions. $\omega_1, \omega_2 \in W$ are said to be equivalent, written $\omega_1 \sim \omega_2$ iff $C_1 \omega_1(\delta) \leqslant \omega_2(\delta) \leqslant C_2 \omega_1(\delta)$ $(0 < \delta \leqslant 1)$.

Theorem 2. *Let $n \in \mathbb{Z}_+$. Then for any $\omega \in W$ there exists a function $f \in C[0, 1]$ such that $\omega_f \sim \omega$ and*

$$\overline{\lim_{q \to \infty}}\, E_1(q) \|p_n(f)_C - p_n(f)_q\|_C > 0, \tag{3}$$

where $E_1(q)$ is the unique solution of (1).

By this theorem estimation (2) is sharp in general for functions with arbitrary moduli of continuity. From Theorems 1 and 2 we obtain following

COROLLARY. *Let* $f \in C[0, 1]$, $\omega_f(\delta) \leqslant \delta^\alpha$ $(0 < \alpha \leqslant 1;\ 0 < \delta \leqslant 1)$, $n \in \mathbb{Z}_+$. *Then for any* $q \geqslant e^\alpha$

$$\| p_n(f)_C - p_n(f)_q \|_C \leqslant C_3(n, f) \frac{\ln q}{q} \tag{4}$$

and for any $0 < \alpha \leqslant 1$ *this order of convergence is in general the best possible.*

Finally, we give a theorem on the uniform convergence of Pólya's algorithm for Lip α.

THEOREM 3. *For any* $n \in \mathbb{Z}_+$, $f \in C[0, 1]$ *with* $\omega_f(\delta) \leqslant \delta^\alpha$ $(0 < \delta \leqslant 1;$ $0 < \alpha \leqslant 1)$ *and* $q \geqslant q_1(n, \alpha)$

$$\| p_n(f)_C - p_n(f)_q \|_C \leqslant C_4(n, \alpha) \left(\frac{\ln q}{q} \right)^{\alpha/(n+\alpha)}, \tag{5}$$

where constants $q_1(n, \alpha)$ *and* $C_4(n, \alpha)$ *depend only on* n *and* α.

PROOF OF THEOREM 1

Let $E_a = E_a(q)$ be the unique solution of the equation

$$\frac{1}{E_a} = \frac{\omega_f(e^{-q/E_a})}{a} \qquad (a > 0, q \geqslant 1), \tag{6}$$

hence $E_1(q)$ defined by (1) equals to $E_1(q)$ defined above. Then evidently for any $q \geqslant 1$

$$\min(1, 1/a)\, E_a(q) \leqslant E_1(q) \leqslant \max(1, 1/a)\, E_a(q), \tag{7}$$

i.e., the solutions of (1) for equivalent moduli are equivalent.

LEMMA 1. *For any* $f \in C[0, 1]$ *such that* $f(s) = 0$ *for some* $s \in [0, 1]$ *and any* $q \geqslant 1$

$$\|f\|_C \leqslant \|f\|_q + \frac{2 \max(1, \omega_f(1))}{E_1(q)}. \tag{8}$$

Proof. We shall consider two cases.

Case 1. $\|f\|_C \leqslant \omega_f(e^{-q})$. Then if $E_1 \leqslant 1$, $\|f\|_C \leqslant \omega_f(e^{-q}) \leqslant \omega_f(1)/E_1$. On the other hand, if $E_1 > 1$, $\|f\|_C \leqslant \omega_f(e^{-q}) \leqslant \omega_f(e^{-q/E_1}) = 1/E_1$. Hence, in this case,

$$\|f\|_C \leqslant \frac{\max(1, \omega_f(1))}{E_1}. \tag{9}$$

Case 2. $\|f\|_C > \omega_f(e^{-q})$. Set $E^*(q) = E_a(q)$, where $a = \|f\|_C$. Then $E^* > 1$. Indeed, if $E^* \leqslant 1$, then by (6)

$$1 \geqslant E^* = \frac{\|f\|_C}{\omega_f(e^{-q/E^*})} \geqslant \frac{\|f\|_C}{\omega_f(e^{-q})} > 1.$$

By this contradiction we obtain, that $E^* > 1$. Further, without loss of generality we may assume that $\|f\|_C = f(s_1)$ and $s_1 > s$. Then obviously $f(x) \geqslant \|f\|_C - \omega_f(s_1 - x)$ for $x \in [s, s_1]$, hence setting $t = \min\{x : \omega_f(x) = \|f\|_C\}$ we obtain

$$\|f\|_q \geqslant \left\{\int_{s_1 - t}^{s_1} (\|f\|_C - \omega_f(s_1 - x))^q \, dx\right\}^{1/q}$$
$$= \left\{\int_0^t (\|f\|_C - \omega_f(x))^q \, dx\right\}^{1/q}. \tag{10}$$

Set now $\bar{t} = \max\{x : \omega_f(x) = \|f\|_C/E^*\}$. Since $E^* > 1$, we have $0 < \bar{t} < t$. This and (10) imply

$$\|f\|_q \geqslant \left\{\int_0^{\bar{t}} (\|f\|_C - \omega_f(x))^q \, dx\right\}^{1/q} \geqslant \left(\|f\|_C - \frac{\|f\|_C}{E^*}\right) \bar{t}^{1/q}. \tag{11}$$

By definition of $\bar{t}$ and E^*,

$$\bar{t} \geqslant e^{-q/E^*}. \tag{12}$$

Further, (7) implies that

$$E^* \geqslant \frac{E_1}{\max(1, 1/\|f\|_C)}.$$

Using this, (12) and (11) we arrive at

$$\|f\|_q \geqslant \|f\|_C \left(1 - \frac{1}{E^*}\right) e^{-1/E^*} \geqslant \|f\|_C \left(1 - \frac{1}{E^*}\right)^2$$
$$\geqslant \|f\|_C - \frac{2\,\|f\|_C}{E^*} \geqslant \|f\|_C - \frac{2\,\|f\|_C \max(1, 1/\|f\|_C)}{E_1}$$
$$\geqslant \|f\|_C - \frac{2\max(1, \omega_f(1))}{E_1}.$$

This inequality together with (9) completes the proof of the lemma.

LEMMA 2. *For any $f \in C[0,1]$ and $q \geqslant 1$,*

$$\|f - p_n(f)_q\|_C \leqslant \|f - p_n(f)_q\|_q + C_5(n)\,\frac{\max(1, \omega_f(1))}{E_1(q)}. \tag{13}$$

Proof. Set $f^*(x) = f(x) - p_n(f, x)_q$; $\bar{f}(x) = f(x) - f(0)$. Since for any polynomial g_n, $\|g_n\|_C \leqslant (2(q+1))^{1/q} n^{2/q} \|g_n\|_q$ (see [10, p. 251]), we have

$$\begin{aligned}\omega_{p_n(f)_q}(\delta) &\equiv \omega_{p_n(\bar{f})_q}(\delta) \leqslant 2n^2\delta \|p_n(\bar{f})_q\|_C \leqslant 2n^2\delta(2(q+1))^{1/q} n^{2/q} \|p_n(\bar{f})_q\|_q \\ &\leqslant 8n^4\delta \|p_n(\bar{f})_q\|_q \leqslant 16n^4\delta \|\bar{f}\|_q \\ &\leqslant 16n^4\delta \|\bar{f}\|_C \leqslant 16n^4\delta\omega_f(1) \leqslant 32n^4\omega_f(\delta).\end{aligned}$$

(In the last inequality we used the fact that for any $0 < \delta_1 \leqslant \delta_2$, $2\omega(\delta_1)/\delta_1 \geqslant \omega(\delta_2)/\delta_2$. See [10, p. 111].) Thus $\omega_{f^*}(\delta) \leqslant C_6(n)\omega_f(\delta)$, where we can put $C_6(n) = 32n^4 + 1$. Further, it is evident that f^* has a zero in $[0,1]$. Thus applying to f^* Lemma 1 we get

$$\|f^*\|_C \leqslant \|f^*\|_q + \frac{2C_6(n)\max(1, \omega_f(1))}{E_{a(n)}(q)},$$

where $a(n) = 1/C_6(n)$. This and (7) imply (13). Q.E.D.

Now we are able to prove Theorem 1. By the strong unicity theorem [9],

$$\|p_n(f)_C - g_n\|_C \leqslant \gamma_n(f)\{\|f - g_n\|_C - \|f - p_n(f)_C\|_C\},$$

where g_n is an arbitrary algebraic polynomial of order at most n. Setting in this inequality $g_n = p_n(f)_q$ and using (13) we obtain the conclusion of Theorem 1.

PROOF OF THEOREM 2

Let $\omega \in W$ be an arbitrary modulus of continuity. Without loss of generality we may asssume that ω is concave and $\lim_{\delta \to +0} \omega(\delta)/\delta > 1$. (Indeed, by a theorem proved in [7] there exists a concave modulus of continuity $\bar{\omega}$ such that $\bar{\omega}/2 \leqslant \omega \leqslant \bar{\omega}$ and multiplying $\bar{\omega}$ by a constant if necessary we can achieve that $\lim_{\delta \to +0} \bar{\omega}(\delta)/\delta > 1$, where $\bar{\omega} \sim \omega$.) Then $\omega(\delta)$ is strictly increasing when $0 < \delta \leqslant \delta_0$ and $\omega(\delta)/\delta$ is decreasing. Therefore the equation

$$\frac{\omega(h_0)}{h_0} = e^{q\omega(h_0)} \tag{14}$$

has a unique solution $h_0 = h_0(q)$ if $q \geqslant q_2(\omega)$.

Assume that $n = 2m$. (The case when $n = 2m + 1$ can be settled similarly.) Set $1/(4m + 4) = b$ and define f on $[0, 4b]$ by

$$
\begin{aligned}
f(x) &= \omega(b) - \omega(b - x), && x \in [0, b];\\
&= \omega(b) - \omega(x - b), && x \in [b, 2b];\\
&= -2\omega(b)\, x/b + 4\omega(b), && x \in [2b, 5b/2];\\
&= -\omega(b), && x \in [5b/2, 7b/2];\\
&= 2\omega(b) x/b - 8\omega(b), && x \in [7b/2, 4b].
\end{aligned}
$$

Extend $f(x)$ to $[0, 1]$ as a $1/(m + 1)$-periodic function. Then evidently $\omega_f \sim \omega$ and $p_n(f)_C \equiv 0$. Set $a_q = \| p_n(f)_q \|_C$, $F_q(x) = |f - p_n(f)_q|^{q-1} \cdot \operatorname{sign}(f - p_n(f)_q)$. By Theorem 1 $a_q \to +0$ as $q \to +\infty$, hence $a_q < \omega(b)$ if $q \geqslant q_8(n, \omega)$. Further by the characterization theorem for best L_q–approximations (see [10, p. 75]) for any $q > 1$

$$\int_0^1 F_q(x)\, dx = 0, \tag{15}$$

i.e.,

$$\int_{f \geqslant 0} F_q(x)\, dx = - \int_{f < 0} F_q(x)\, dx. \tag{16}$$

Let us estimate these integrals.

$$
\begin{aligned}
\int_{f \geqslant 0} F_q(x)\, dx &\leqslant \int_{f \geqslant 0} |f - p_n(f)_q|^{q-1}\, dx\\
&\leqslant \int_{f \geqslant 0} (f + a_q)^{q-1}\, dx \leqslant \frac{1}{2b} \int_0^b (f + a_q)^{q-1}\, dx. \qquad (17)
\end{aligned}
$$

Let $h_0 = h_0(q)$ be the unique solution of (14). Then using concavity of $\omega(\delta)$ we have

$$
\begin{aligned}
&\int_0^b (f + a_q)^{q-1}\, dx\\
&\quad \leqslant \int_0^{b-h_0} (f + a_q)^{q-1}\, dx + \int_{b-h_0}^b (f + a_q)^{q-1}\, dx\\
&\quad \leqslant \int_0^{b-h_0} \left(\frac{\omega(b) - \omega(h_0)}{b - h_0} x + a_q \right)^{q-1} dx\\
&\qquad + \int_{b-h_0}^b \left\{ \frac{\omega(h_0)}{h_0} x + \omega(b) - \frac{b\omega(h_0)}{h_0} + a_q \right\}^{q-1} dx\\
&\quad \leqslant \frac{b - h_0}{\omega(b) - \omega(h_0)} \frac{(\omega(b) - \omega(h_0) + a_q)^q}{q} + \frac{h_0}{\omega(h_0)} \frac{(\omega(b) + a_q)^q}{q}
\end{aligned}
$$

$$\leqslant \frac{C_7(\omega)}{q} \{(\omega(b) - \omega(h_0) + a_q)^q + (e^{-\omega(h_0)}(\omega(b) + a_q))^q\}$$

$$\leqslant \frac{C_7(\omega))}{q} \left\{(\omega(b) + a_q - \omega(h_0))^q + (\omega(b) + a_q - \frac{\omega(b)}{2} \omega(h_0))^q\right\}$$

$$\leqslant \frac{2C_7(\omega)}{q} (\omega(b) + a_q - C_8(n, \omega)\omega(h_0))^q,$$

where $C_8(n, \omega) = \min\{1, \omega(b)/2\}$ and $h_0(q)$ is small enough $(q \geqslant q_3(n, \omega))$. This and (17) imply

$$\int_{f \geqslant 0} F_q(x)\, dx \leqslant \frac{C_9(n, \omega)}{q} (\omega(b) + a_q - C_8(n, \omega)\omega(h_0))^q. \tag{18}$$

Now we shall give a lower estimation for $-\int_{f<0} F_q(x)\, dx$.

$$-\int_{f<0} F_q(x)\, dx$$

$$= -\int_{f \leqslant -a_q} F_q(x)\, dx - \int_{-a_q \leqslant f < 0} F_q(x)\, dx$$

$$\geqslant \int_{f \leqslant -a_q} (-f - a_q)^{q-1}\, dx - \int_{-a_q \leqslant f < 0} (-f + a_q)^{q-1}\, dx$$

$$\geqslant \frac{1}{4b} \int_{5b/2}^{7b/2} (\omega(b) - a_q)^{q-1}\, dx - (2a_q)^{q-1}$$

$$= \frac{1}{4} (\omega(b) - a_q)^{q-1} - (2a_q)^{q-1} \geqslant \frac{1}{5} (\omega(b) - a_q)^{q-1} \qquad (q \geqslant q_4(n, \omega)).$$

Combining this inequality with (16) and (18) we obtain

$$(\omega(b) + a_q - C_8(n, \omega)\, \omega(h_0))^q$$
$$\geqslant C_{10}(n, \omega)\, q(\omega(b) - a_q)^{q-1} \geqslant C_{11}(n, \omega)\, q(\omega(b) - a_q)^q.$$

Thus

$$\omega(b) + a_q - C_8(n, \omega)\, \omega(h_0) \geqslant (\omega(b) - a_q)(C_{11}(n, \omega) q)^{1/q}$$

$$\geqslant (\omega(b) - a_q) \left(1 + \frac{\ln C_{11}(n, \omega) q}{q}\right) = \omega(b) - a_q$$

$$+ \omega(b) \frac{\ln C_{11}(n, \omega) q}{q} - a_q \frac{\ln C_{11}(n, \omega) q}{q},$$

i.e.,

$$a_q \geqslant C_{12}(n, \omega)\left(\omega(h_0) + \frac{\ln q}{q}\right) \qquad (q \geqslant q_5(n, \omega)). \tag{19}$$

Let us consider two cases.

Case 1. There exists a sequence of positive numbers $\{\delta_k\} \to +0$ such that $\omega(\delta_k) > \sqrt{\delta_k}$.

Let E^* be the unique solution of the equation

$$\frac{1}{E^*} = \omega(e^{-q/E^*}).$$

Equivalence of ω_f and ω implies that $C_{13}(n, \omega)\, E^*(q) \leqslant E_1(q) \leqslant C_{14}(n, \omega)\, E^*(q)$, where $E_1(q)$ is the unique solution of (1). Set $1/E^* = \omega(h_1)$. If q is big enough then $h_1 = h_1(q)$ satisfies the relation

$$\ln \frac{1}{h_1} = q\omega(h_1) \tag{20}$$

and $h_1 > h_0$. We can choose a sequence $q_k \to +\infty$ satisfying $h_0(q_k) = \delta_k$. Thus by (14) and (20)

$$\begin{aligned} q_k\omega(h_0(q_k)) &= \ln \frac{\omega(h_0(q_k))}{h_0(q_k)} = \ln \frac{\omega(\delta_k)}{\delta_k} > \frac{1}{2} \ln \frac{1}{\delta_k} \\ &= \frac{1}{2} \ln \frac{1}{h_0(q_k)} > \frac{1}{2} \ln \frac{1}{h_1(q_k)} = \frac{1}{2}\, q_k\omega(h_1(q_k)) \qquad (k \geqslant k_0), \end{aligned}$$

i.e.,

$$\omega(h_0(q_k)) > \frac{1}{2}\,\omega(h_1(q_k)) = \frac{1}{2E^*(q_k)} \geqslant \frac{C_{15}(n, \omega)}{E_1(q_k)} \qquad (k \geqslant k_0).$$

Then by (19)

$$\| p_n(f)_C - p_n(f)_{q_k} \|_C \geqslant \frac{C_{16}(n, \omega)}{E_1(q_k)} \qquad (k \geqslant k_0)$$

hence (3) is verified in this case.

Case 2. Let us consider the opposite case. Then $\omega(\delta) \leqslant \sqrt{\delta}$ $(0 < \delta \leqslant \delta_1)$. But this implies that $1/E_1(q) \leqslant C_{17}(n, \omega) \ln q/q$. Thus using (19) we have

$$\| p_n(f)_C - p_n(f)_q \|_C = a_q \geqslant C_{12}(n, \omega) \frac{\ln q}{q} \frac{C_{18}(n, \omega)}{E_1(q)} \qquad (q \geqslant q_6(n, \omega)),$$

which verifies (3) in Case 2.

The proof of Theorem 2 is completed.

PROOF OF THEOREM 3

Let $f \in C[0, 1]$ and $\omega_f(\delta) \leqslant \delta^\alpha$ $(0 < \alpha \leqslant 1; 0 < \delta \leqslant 1)$. Then by Lemma 2

$$\|f - p_n(f)_q\|_C \leqslant \|f - p_n(f)_q\|_q + \frac{C_5(n)}{\alpha} \frac{\ln q}{q}$$

$$\leqslant \|f - p_n(f)_C\|_C + \frac{C_5(n)}{\alpha} \frac{\ln q}{q} \qquad (q \geqslant e^\alpha). \tag{21}$$

Further, we shall need the following result: for any $0 < \varepsilon \leqslant 1$ and $0 < \alpha \leqslant 1$

$$\sup_{f: \omega_f(\delta) \leqslant \delta^\alpha} \sup_{\substack{g_n \in \Pi_n \\ \|f - g_n\|_C \leqslant \|f - p_n(f)_C\|_C + \varepsilon}} \| p_n(f)_C - g_n \|_C \leqslant C_{19}(\alpha, n) \varepsilon^{\alpha/(n+\alpha)}, \tag{22}$$

where Π_n is the set of algebraic polynomials of order at most n and $C_{19}(\alpha, n)$ depends only on n and α. Equation (22) was essentially proved in [8] because it easily follows from Lemmas 2, 3 and 5 of [8]. We shall outline the proof. By Lemmas 2 and 3 of [8] if $f \in C[0, 1]$ satisfies $\omega_f(\delta) \leqslant \delta^\alpha$ and $0 \leqslant x_0^{(n)} < x_1^{(n)} \cdots < x_{n+1}^{(n)} \leqslant 1$ are its points of Tchebycheff deviation (that is, $(f - p_n(f)_C)(x_i^{(n)}) = \gamma(-1)^i \|f - p_n(f)_C\|_C$, $\gamma = \pm 1$; $i = 0, 1, ..., n+1$), then $x_{i+1}^{(n)} - x_i^{(n)} \geqslant C_{20}(n, \alpha) \|f - p_n(f)_C\|_C^{1/\alpha}$, $i = 0, 1, ..., n$. By Lemma 5 of [8] if $\bar{g}_n \in \Pi_n$ satisfies relations $\bar{\gamma}(-1)^{i+1} \bar{g}_n(x_i) \leqslant \mu$ $(\bar{\gamma} = \pm 1;$ $\mu > 0;$ $i = 0, 1, ..., n+1)$, where $0 \leqslant x_0 < \cdots < x_{n+1} \leqslant 1$ and $x_{i+1} - x_i \geqslant \lambda > 0$ $(i = 0, 1, ..., n)$, then $\| \bar{g}_n \|_C \leqslant C_{21}(n) \mu / \lambda^n$. Take now arbitrary $g_n \in \Pi_n$ satisfying $\|f - g_n\|_C \leqslant \|f - p_n(f)_C\|_C + \varepsilon$. Then it is easy to see that $\gamma(-1)^{i+1}(g_n - p_n(f)_C)(x_i^{(n)}) \leqslant \varepsilon$ $(i = 0, 1, ..., n+1)$ hence by previous remarks

$$\| p_n(f)_C - g_n \|_C \leqslant \frac{C_{22}(n, \alpha) \varepsilon}{\|f - p_n(f)_C\|_C^{n/\alpha}}.$$

If $\|f-p_n(f)_C\|_C > \varepsilon^{\alpha/(n+\alpha)}$ then this implies that $\|p_n(f)_C - g_n\|_C \leqslant C_{22}(n, \alpha)\varepsilon^{\alpha/(n+\alpha)}$. If, on the contrary, $\|f-p_n(f)_C\|_C \leqslant \varepsilon^{\alpha/(n+\alpha)}$, then

$$\begin{aligned}\|p_n(f)_C - g_n\|_C &\leqslant \|f-p_n(f)_C\|_C + \|f-g_n\|_C \\ &\leqslant 2\|f-p_n(f)_C\|_C + \varepsilon \leqslant 2\varepsilon^{\alpha/(n+\alpha)} + \varepsilon \leqslant 3\varepsilon^{\alpha/(n+\alpha)}.\end{aligned}$$

Thus the proof of (22) is completed.

Using (22) and (21) we immediately obtain (5). Q.E.D.

Remark. A detailed proof of (22) in the periodic case can be found in [11].

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